Problem Solving Group

Problem Set 11: Last PSG (special problems)

May 1, 2025

Please read each problem carefully. Otherwise, Rowan will be sad, which adds another problem to the already lengthy list of problems (below). If you have any problems with this problem set, please format them sequentially as "[$k+1$]. [your problem here]", where k is the number of problems currently on the problem set.

Problems

1. "AAAAAAAAGH!! MY COMPUTER!!!" Ashley screams as a virus suddenly begins destroying her precious files. Jesse appears in the corner of the room, laughing maniacally. The virus he had somehow installed on her computer was destroying $1/2^t$ of her remaining data per second, where t is initially 1, and increments by 1 every second. What proportion of her data will not be destroyed, assuming the virus is unstoppable?

2. The probability P that Sam reads this very problem is initially $1/2$, as he may or may not come to PSG. However, for every individual who is neither Sam nor Rowan who reads this problem, either (A) P increases by $1/8$, as that individual may be inclined to tell Sam to read the problem; or (B) P is cut in half, as the individual may feel strongly about not telling Sam about the problem unless he comes to PSG to see for himself. Let P_t denote the value of P after t non-Sam-non-Rowan people have read the problem, so i.e. $P_0 = 1/2$, P_1 is either $5/8$ or $1/4$, etc. Suppose that for each person who reads this problem, event (A) occurs with probability $1 - P_t$, and event (B) occurs with probability P_t.

    (a) Determine the value of P_t such that $E(P_{t+1}) = P_t$.

    (b) What is the expected value of P_∞?

3. For a brief moment, Adam's voice can be heard from afar.

4. Thea bakes two different types of cookies, and arranges them all in an $n \times m$ grid. (You do not need to reenact this.) Determine the total number of ways to remove cookies from the grid such that (1) the first cookie removed is the bottom-left one, (2) the last cookie is on the top right, and (3) every cookie removed must be adjacent (horizontally or vertically) to the previously-removed cookie. Note: the minimum number of cookies removed to make such a path (bottom left to top right) is $n+m-1$, whereas the maximum is $\sim nm$.

5. Sometime in the future, Julia decides to run a 10k. During this race, her speed in mph x hours into the race is given by

$$f(x) = e^{-x^2(x-1)^2} + 7.$$

    Determine the amount of time it took her to run the 10k (6.25 miles).

6. Jack incorrectly munches on a banana over the course of 10 minutes. A standard banana contains 80 calories, and it typically takes Jack 10 calories to chew and metabolize one banana. Furthermore, Jack has a base metabolic rate of 90 calories per hour, and exactly half of the time he is eating the banana, he also spends continuously doing jumping jacks, which burns 12 calories per minute. What is the net change in metabolic energy of Jack from the time he begins eating the banana to the time he finishes?

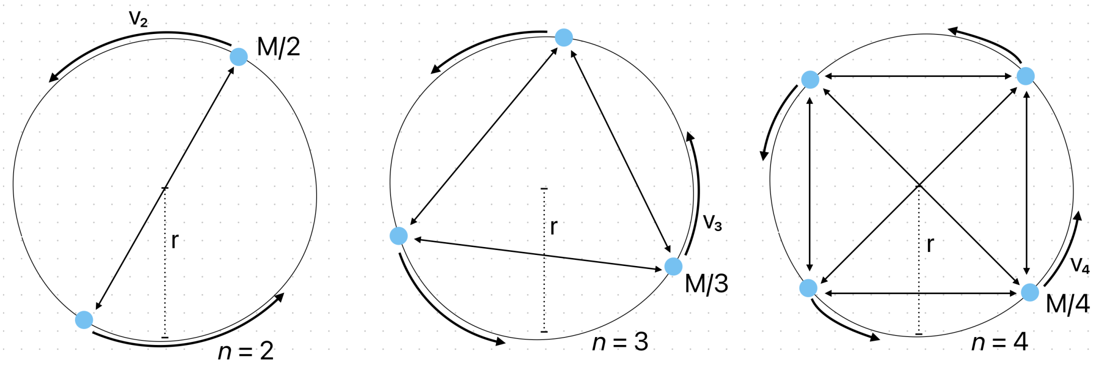
7. I couldn't think of a 7th problem.

8. Every 30 minutes until eternity, Conner does n pull-ups on one of the door frames in the math lounge, where n is a randomly selected number ranging from 2 to 6. Each pull-up is associated with odds of 4 thousand to 1 that Conner will break something. What is the expected amount of time Conner must do pull-ups for before he breaks something?

9. If you are Jesse, do not read problem 1.

10. Ethan is wondering how many integers x between 1 and 2025 (inclusive) have the property that $x = x^3 \mod 5$. (Don't ask me why.)

11. Kaiben is considering a specific instance of the three-body problem, in which a total mass of M is distributed evenly across three stars, which are all equally spaced around a ring of unchanging radius r, orbiting each other. See middle figure, below.



   (a) Given that the force of gravity between masses m_1 and m_2 separated by distance d is $\frac{Gm_1m_2}{d^2}$, and that the centripetal force required to keep a mass-m object in an orbit of radius r is $\frac{mv^2}{r}$, where v is the orbital speed, determine the orbital speed of this three body system in terms of G, M, and r. (Hint: set net gravitational force equal to centripetal force.)

   (b) Consider the same scenario, but for 4 identical stars orbiting each other around the same ring, again with the total mass of the system being M. Solve for v.

   (c) Determine v_n, the orbital speed of each body in an n-body system of this type.

   (d) Compute $\lim_{n\to\infty} v_n$.

12. Ben and Kevin decide to make some investments. Ben invests \$100 into the dehydrated water industry, seeing as it returns about 5% annually, and adds \$10 to his account every year. Kevin invests \$500 into pet rock production, and gets a rate of r returned annually. He also adds \$10 to his account every year.

   (a) What must the value of r be in order for Kevin and Ben to have exactly the same amount of money in their accounts after 100 years?

   (b) Iris decides to start investing also. Iris is smart about it though, and invests \$100 in a startup company that produces what critics are referring to as Hagoromo 2.0. The return rate is initially a, which increases at a constant rate of 0.1% annually. Assuming that Iris also deposits an additional \$10 per year, what must a be in order for the amount produced after 100 years to be the same as for Ben and Kevin?

13. I think Will may have a problem of his own to pose.

14. Rowan wants to solve problem 14, but he cannot. Why?

15. All of the seniors are graduating, which is a problem! Thus, it is up to you to spread the word about PSG!!

References

Brain.