An assortment of mathematical insights

by Rowan Shigeno

Last updated: 2/3/2026

Introduction.

The problem-solution pairs and derivations that I chose to compile here can be described as having at least one of the following attributes:

1. The problem posed comes across as simple, but the solution is surprisingly not so simple.
2. The problem is seemingly complex, but the solution is unexpectedly quite simple.
3. The process in which we obtain the solution is elegant, perhaps involving an area of mathematics or physics which we do not think to be at all relevant.
4. It's just kinda neat, dude.

The explanations below, furthermore, are generally written under the assumption that the reader is at least fairly well-versed in calculus, and is best suited to individuals who have some appreciation for mathematical concision and elegance. If you do not, what follows will (hopefully) convince you why you should.

Contents.

1 An undead population growth problem.

Consider the following scenario: an undead spawns into existence. At the end of the day, it either (a) vanishes, (b) does nothing, (c) revives another undead, or (d) revives two undead, all with equal probability. Any subsequent undead are subject to the same laws of probability at the end of each day. The question: *can we expect this process to continue forever, or for all undead to cease to exist after some finite amount of time?*

To answer this question, consider a real-world scenario in which an individual (or pair of individuals) has some number of offspring (0, 1, 2, etc.). If the expected (average) number of offspring each individual or pair of individuals have is the same as the number of parents, then the population is stable, and neither increases nor decreases in the long term; but if on average the number of offspring is 3, for instance, the population will grow exponentially.

The same logic applies here, with the exception that the original "parent" undead never dies: and so, if we find on average that the undead revives no one else, then we cannot conclude either way whether the population will go extinct or not. The expected number of zombies after each cycle is given by $\frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{4}(2) + \frac{1}{4}(3) = \frac{3}{2}$. Since $\frac{3}{2} > 1$, we can expect there to be 50% more zombies each day, and therefore conclude that the undead will (probably) not go out of existence.

Now to arrive at a more precise conclusion, we can take x to be the probability that the population goes extinct. From the start, we immediately see that x is at least $\frac{1}{4}$; if the undead does nothing, the probability is then $1/4x$; but if it revives another, the probability $1/4x^2$ (due to there now being two undead), and if it revives two, this is $1/4x^3$ (three undead). Thus in total, we have

$$x = \frac{1}{4} + \frac{1}{4}x + \frac{1}{4}x^2 + \frac{1}{4}x^3.$$

Solving for x yields $\sqrt{2} - 1$, or approximately 41%: these less-than-even odds agree with our initial conclusion that the population will most likely not go to zero.

2 The Brachistochrone problem.

The Brachistochrone curve is the surface for which a ball, under the influence of gravity, will roll frictionlessly between two points in the minimal amount of time. Surprisingly, this is not a straight line: although this certainly yields the least *distance* between points, it does not provide the least *time*. By the law of conservation of energy, we know that $\frac{1}{2}mv^2 = mgy$, and so

$$v = \frac{ds}{dt} = \sqrt{2gy}$$

Meaning that speed is independent of horizontal displacement (again, ignoring friction). Assume, for simplicity, the particle departs from the origin $(0, 0)$, and reaches maximum speed v_m after falling some vertical distance D: that is, $v_m = \sqrt{2gD}$.

Now while we might attempt to minimize time using calculus, Fermat's principle— which states that the path taken by a ray of light between two points will be the one that takes the least amount of time— is coincidentally applicable to this problem. Due to Snell's law (a consequence of this principle), light passing from one medium to another will refract such that

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2},$$

where θ_1 is the angle of incidence, θ_2 is the angle of refraction, v_1 and v_2 correspond to the relative phase velocities.

Instead of a medium or a sequence of mediums, however, we can imagine our rolling ball to be subject to the continuous acceleration due to gravity. At the endpoint of the ball's rolling— when velocity is maximized— notice that $\sin\theta$ can at most be 1, indicating that its direction of travel at this point corresponds to the angle $\frac{\pi}{2}$ (horizontal): thus we write

$$\frac{\sin\theta}{v} = \frac{1}{v_m},$$

where v and θ are the ball's speed and direction of travel at any given moment between the time it is dropped and the time it arrives (where the curve levels out horizontally). Observe also that at the onset, we have $v = 0$, and so $\theta = 0$ in order for $v_2 \sin\theta_1 = v_1 \sin\theta_2$ to hold true.

Now substituting $\sin\theta = \frac{dx}{ds}$ into the above equation,

$$v_m dx = v ds.$$

Since $ds^2 = dx^2 + dy^2$,

$$v_m^2 dx^2 = v^2(dx^2 + dy^2) \quad \longrightarrow \quad v^2 dy^2 = v_m^2 dx^2 - v^2 dy^2.$$

Solving for $\frac{dy}{dx}$,

$$\frac{dy^2}{dx^2} = \frac{v_m^2 - v^2}{v^2} = \frac{2gD - 2gy}{2gy} = \frac{D-y}{y} \quad \longrightarrow \quad \boxed{\frac{dy}{dx} = \sqrt{\frac{D-y}{y}}}.$$

Solving this differential equation results in an inverted cycloid generated by a circle of diameter D.

Additionally, it can be shown that the rate of change of direction of the bead with respect to time, $\frac{d\theta}{dt}$, is in fact constant. Since $\frac{dv}{dt} = \frac{d^2s}{dt^2} = g\cos\theta$,

$$\frac{d}{dt}\sin\theta = \frac{d}{dt}\frac{v}{v_m} = \frac{d^2s}{dt^2}\frac{1}{v_m} = \frac{g}{v_m}\cos\theta;$$

But because $\frac{d}{dt}\sin\theta = \frac{d\theta}{dt}\cos\theta$, $\frac{g}{v_m}\cos\theta = \frac{d\theta}{dt}\cos\theta$, i.e. $\boxed{\frac{d\theta}{dt} = \frac{g}{v_m}}$.

3 The n-village road-building problem.

Consider the following scenario: there are 4 villages positioned at the corners of a square, each side one mile in length. We want to build a series of roads between the villages such that one can access any village from any other village, minimizing the total road length (or road material used).

A first guess would be to just connect the four villages by the edges of the square: here, the total road length would be 4 miles. We can in fact remove one of the four roads (say, the top of the square) and still have all 4 villages be connected in some way, yielding a total of 3 miles. To distribute travel distance between villages as fairly as possible, we can move the translate the bottom road upwards, to the center, to make an 'H'.

An alternative, close-to-optimal solution would be to link the 4 villages together via the diagonals of the square, making an 'X'. The total distance here is $2\sqrt{2} \approx 2.83$. But there is an even better solution.

Consider taking the 'H' diagram, but contracting the intersection points (the midpoints of the left and right roads) to the center of the square, as follows:

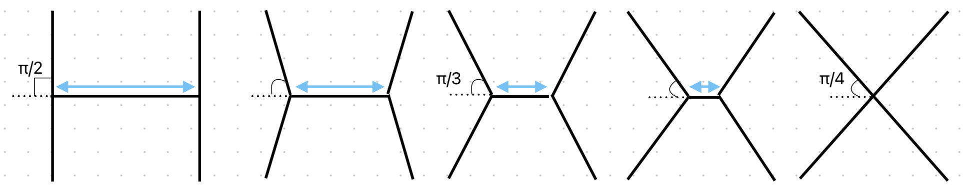


Figure 1: contraction of H to X.

Keeping track of the labeled angle, we can write the total road length L in terms of the angle, using trigonometry:

$$L = \frac{2}{\sin\theta} + 1 - \frac{1}{\tan\theta}.$$

This quantity is minimized when $\theta = \frac{\pi}{3}$, making $L \approx 2.73$. This is the optimal road construction.

This problem can also be solved intuitively by making it an energy-minimization problem. Considering instead in the above diagram the roads to be ropes (with no slack) having some tension T, the tension vectors, which point away from each point the two points of intersection, only cancel out perfectly (so that $F_{net} = 0$) when $\theta = \frac{\pi}{3}$.

4 Factors of 2 in $2^k!$

How many factors of two are in the prime factorization of 2^k factorial?

Recall that $2^k!$ is the product of all integers from 1 through $2^k!$. Observe first that only half of these integers are even (i.e., contain at least one factor of 2); further, exactly half of these even integers are simply two times an odd number (i.e., contain only one factor of 2). Half of the remaining evens contain 3 factors of two, half of the other half contain 4, and so on. Thus, the total number of powers of two is

$$2^k(\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \cdots + \frac{k-1}{2^k} + \frac{k}{2^k})$$

Which is equivalent to

$$2^{k-1}\sum_{n=1}^{k-1}\frac{n}{2^n} + k.$$

Now notice that $\sum_{n=1}^{\infty} nx^n$ is the Taylor series for $\frac{x}{(x-1)^2}$. Taking $x = 1/2$ yields that $\sum_{n=1}^{\infty}\frac{n}{2^n} = \frac{1/2}{1/4} = 2$, and with some algebra we find that $\sum_{n=1}^{k-1}\frac{n}{2^n} = 2 - \frac{k-1}{2^{k-1}}$. Thus, the above expression is equal to $\boxed{2^k - 1}$.

For example: consider $2^3! = 8!$. Clearly, the powers of two here come from the factors $2, 4, 6, 8$, which yield a total of $1 + 2 + 1 + 4 = 7$ powers of 2. Indeed, $2^3 - 1 = 7$.

A larger example: $2^7! = 128!$. We have that $2^7 - 1 = 127$ and so $2^7!$ should be divisible by 2^{127}, but not 2^{128}. Check: $2^7!/2^{127} \mod 10 = 5$.

5 A probabilistic argument for the Collatz conjecture.

The Collatz conjecture is a famous unsolved problem in mathematics. It asks whether, given any positive integer n, n will always eventually reach 1 in a finite number of steps via the Collatz map. For each step in this process, we take an input n, mapping it to $n/2$ if n is even, and $3n+1$ if n is odd —and repeat with the result.

The conjecture was initially introduced by Lothar Collatz in 1937. Despite there being hundreds of published papers on the conjecture, and many unpublished failed attempts at proofs, the conjecture remains unsolved. However, we do have some partial results: in particular, a 2017 distributed computing project verified the Collatz conjecture for all starting values of n up to 10^{20}. Furthermore, it was shown in 1993 that any nontrivial "cycle" of numbers obtained via Collatz (i.e. a group of numbers that form a loop and thus never descent to 1) must have length at least 17,087,915! Thus, simply finding a counterexample is extremely unlikely.

Despite the remaining uncertainty regarding when (if ever) this conjecture will be proven (or disproven), we can construct a somewhat convincing heuristic argument for why the Collatz conjecture is *probably* true. Observe:

1. 2^n takes exactly $n-1$ steps to reach 1.

2. $3n+1$ is always even for n odd, whereas $n/2$ could be even or odd for n even. Thus, we often observe long sequences of division by two, but we can never have consecutive $3n+1$-steps. As a result, the maximum rate at which a chain can grow is approximately $1.5n$ (every 2 steps).

3. For notational convenience, let X represent the event that we take $n/2$, and Y represent the event of taking $3n+1$. Given some large even number N s.t. $2^k \leq N < 2^{k+1}$, notice that $P(X^k) = 1/2^{k-1}$ and $P(X^{k-1}) = 1/2^{k-1}$, as 2^k and $3 \cdot 2^{k-1}$ are two of the 2^{k-1} even numbers in this interval. $P(X^{k-2}) = 2/2^{k-1} = 1/2^{k-2}$, $P(X^{k-3}) = 4/2^{k-1} = 1/2^{k-3}$, $P(X^{k-4}) = 8/2^{k-1} = 1/2^{k-4}$, and so on, until $P(X^1) = 1/2$ (since half of the even numbers are simply two times an odd numbers). Thus the expected number of divisions by two of N is

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots + \frac{k-1}{2^{k-1}} + \frac{k}{2^{k-1}}.$$

This quantity is approximately equal to 2 for k very large (see previous section), so we can observe that for large values of n, we expect n to be reduced to $0.75n$ three steps later (multiply by 3, divide by 2 twice). And since $0.75 < 1$, numbers should *on average* decrease by 25%, and thus eventually arrive at 1.

6 Generating Pythagorean triples.

Take the complex number $z = a + bi$. Square it: then the resulting number is $z^2 = (a + bi)^2 = a^2 - b^2 + 2abi$. Now notice that the modulus of z^2 is the square root of $(a^2 - b^2)^2 + 4a^2b^2 = a^4 + 2a^2b^2 + b^4 = (a^2 + b^2)^2$: that is, $|z^2| = a^2 + b^2$. As a result, we find that the following set of numbers is guaranteed to result in a Pythagorean triple, given positive integers a, b:

$$(a^2 - b^2, 2ab, a^2 + b^2).$$

7 Taylor series and Euler's formula.

Recall that the sum of a geometric series with initial term a and common ratio r with $-1 \leq r \leq 1$ is $\frac{a}{1-r}$. Thus, we can in fact write the the function $\frac{1}{1-x}$ as being the sum of infinite series $\sum_{n=1}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$, provided that $-1 \leq x \leq 1$.

This relationship can in turn be manipulated (using substitution, differentiation, or integration) to represent desired functions, which can be use to make approximations. For instance, we integrate and obtain

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

And plug in $x = -1$ and multiply by -1 to obtain the famous series representation for $\ln 2$:

$$-\ln(2) = \sum_{n=1}^{\infty} -\frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Now to generalize these "power series" to any function, we assume $f(x)$ has representation $a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$, and repeatedly differentiate to find that $f(c) = a_c$,$f'(c) = a_1$,$f''(c) = 2a_2$, $f'''(c) = 6a_3$, and in general $f^{(n)}(c) = n!a_n$. Thus, if we let $a_n = \frac{f^{(n)}(c)}{n!}$, we have that

$$f(x) = \sum_{n=1}^{\infty} a_n(x-c)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

This is called the *Taylor series* of f about c; when $c = 0$, the result is the *Maclaurin series*. In the particularly interesting case of $f(x) = e^x$ (for which $f'(x) = f(x)$), the n^{th} coefficient is the inverse of n factorial:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \cdots$$

If we apply the same process to $\sin x$ and $\cos x$, we find that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots .$$

These representations looks somewhat similar to that of e^x. In fact, if we substitute ix for x in e^x, we find that $\boxed{e^{ix} = \cos x + i \sin x}$. This is known as *Euler's formula*, and the specific case in which $x = \pi$, yielding $e^{\pi i} = -1$, is known as *Euler's identity.* Euler's formula also provides an alternate proof of DeMoivre's theorem: for complex number $z = r(\cos\theta + i\sin\theta)$, we have that

$$z^n = [r(\cos\theta + i\sin\theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta).$$

Note: the formula $e^{ix} = \cos x + i \sin x$ can also be verified via differentiation. Letting $f(x) = \cos x + i \sin x$, we have that $f'(x) = -\sin x + i\cos x = i(\cos x + i \sin x) = if(x)$. But the only function for which $f'(x) = if(x)$ is e^{ix}: hence, $f(x) = e^{ix}$.

8 Product of sines from $\theta = \frac{k\pi}{n}$ for $k = 1$ to $n - 1$.

Evaluate the product

$$\sin\frac{\pi}{11}\sin\frac{2\pi}{11}\cdots\sin\frac{10\pi}{11}$$

exactly.

To approach this, consider first a few simpler cases. For instance, it is not hard to see that

$$\sin\frac{\pi}{3}\sin\frac{2\pi}{3} = \frac{3}{4},$$
$$\sin\frac{\pi}{4}\sin\frac{\pi}{2}\sin\frac{3\pi}{4} = \frac{1}{2},$$
$$\sin\frac{\pi}{6}\sin\frac{2\pi}{6}\sin\frac{3\pi}{6}\sin\frac{4\pi}{6}\sin\frac{5\pi}{6} = \frac{3}{16}.$$

For the product of $\sin\frac{k\pi}{8}$, utilize the fact that $\sin(\frac{\pi}{2} - \theta) = \cos\theta$ and the identity $\sin 2\theta = 2\sin\theta\cos\theta$:

$$\sin\frac{\pi}{8}\sin\frac{\pi}{4}\cos\frac{\pi}{8}\sin\frac{\pi}{2}\sin\frac{\pi}{8}\sin\frac{\pi}{4}\cos\frac{\pi}{8} = (\sin\frac{\pi}{8}\cos\frac{\pi}{8})^2(\sin\frac{\pi}{4})^2$$
$$= \frac{1}{4}\sin^4\frac{\pi}{4} = \frac{1}{16}.$$

And in general, we notice a pattern:

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}.$$

To prove that this pattern holds for all n, we utilize the exponential formula for $\sin x$: that is, $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$. Then we can write

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \prod_{k=1}^{n-1} \frac{1}{2i}(e^{\frac{k\pi i}{n}} - e^{-\frac{k\pi i}{n}}) = (\frac{1}{2i})^{n-1} \prod_{k=1}^{n-1} e^{\frac{k\pi i}{n}} \prod_{k=1}^{n-1}(1 - e^{-\frac{2k\pi i}{n}}).$$

The first product evaluates to

$$\prod_{k=1}^{n-1} e^{\frac{k\pi i}{n}} = e^{\sum_{k=1}^{n-1} \frac{k\pi i}{n}} = e^{\frac{(n-1)\pi i}{2}} = (e^{\frac{\pi i}{2}})^{n-1} = i^{n-1},$$

And the second give us the non-trivial roots of unity, simplifying to n. Thus, the expression reduces to

$$(\frac{1}{2i})^{n-1} \cdot i^{n-1} \cdot n = \boxed{\frac{n}{2^{n-1}}}.$$

9 Connectedness of k^{th}-order Voronoi cells.

We define a *Voronoi diagram* for a set of sites S to be a collection of $|S|$ cells such that for any point p in cell $V(s)$ ($s \in S$), $d(p, s) < d(p, s_i)$ for all $s_i \in S$ ($s_i \neq s$). A *k^{th}-order Voronoi diagram* is a collection of cells such that each cell corresponds to all points that are closer to k given sites of S than any other points in S. For example, $V(a, b, c)$ is the 3rd-order Voronoi cell consisting of all points whose 3 closet sites are a, b and c.

Theorem. *$V\{s_1, \cdots, s_j\}$, i.e. any Voronoi cell of any order, is connected.*

Proof. Assume for the sake of contradiction that $V\{s_1, \ldots, s_j\}$ is the union of two disjoint sets A, B, i.e. $V(Q) = A \cup B$, where $Q = \{s_1, \ldots, s_j\}$. Notice that all points in A must lie strictly to the "left side" of all bisectors which correspond to the directed edges $J_1, \ldots, J_k$ (pointing counterclockwise around A) that make up the border of A, i.e.

$$\partial A = \bigcup_{i=1}^{k} J_i \qquad \text{and} \qquad A = \bigcap_{i=1}^{k} J_i^-,$$

Where we say J_i^- is the entire region on the left side of edge J_i. Since exiting A requires crossing such a bisector, all points outside of A lie on the "right side" of at least one bisector— that is, in the region J_i^+ for the corresponding edge J_i. Now choose some edge $J \subset a \perp b$ of A such that (1) $a \in Q$ (which implies $b \notin Q$), and (2) some subset $B' \subseteq B$ lies on the opposite side of $a \perp b$ than A, i.e. $B' \in J^+$. Observe that we can always find such a J because B is entirely in the complement of A, and so every point in B lies to the right of at least one of the edges in $\{J_1, \ldots, J_k\}$.

Now since by definition of bisection $d(c, a) < d(c, b)$ for any $c \in J^-$ and $d(d, b) < d(d, a)$ for any $d \in J^+$, we have that all points in B' are closer to b than to a. But since $B' \subset V(Q)$, $a \in V(Q)$, and $b \notin V(Q)$, this is a contradiction. Thus, $V(Q)$ must be connected. □

10 Convergence of a rose process to a point.

In some metric, the *centroid* of a set of points is the "average" position of all points. This can be easily computed in Euclidean by taking the average of the Cartesian coordinates of these points. In other metrics, however, this is not so simple.

We specifically examine how one might go about at least *approximating* a centroid (specifically, a projectively invariant one) in a metric called the *Hilbert metric*, essentially a generalization to hyperbolic space to those having polygonal-boundary borders. To determine the distance between two points p, q in Hilbert, we construct a straight line intersecting p and q, and let the intersection of the line with the boundary Ω be the points x, y, so that the points are lined up as x, p, q, y. Then distance is defined as

$$d(p, q) = \frac{1}{2} \ln \frac{\|p - y\|}{\|q - y\|} \cdot \frac{\|q - x\|}{\|p - x\|}.$$

Now to determine the "centroid" of a set of points $P = \{p_0, p_1, \ldots, p_n\}$ in convex position, we do the following:

1. Let $R_0 = P$.
2. For each $i > 0$, $R_i = \{\text{mid}(p'_0, p'_1), \text{mid}(p'_1, p'_2), \ldots, \text{mid}(p'_n, p'_0)\}$ where $R_{i-1} = \{p'_j\}$ for $j = \{0, 1, \ldots, n\}$.
3. Call the point $C = \lim_{i\to\infty} R_i$ the *rose-centroid* of S.

So e.g. $R_2 = \{\text{mid}(\text{mid}(p_0, p_1), \text{mid}(p_1, p_2)), \ldots, \text{mid}(\text{mid}(p_n, p_0), \text{mid}(p_0, p_1))\}$.

Theorem. *The rose-centroid of a set of convex points is a single point.*

Proof. We want to prove that the rose process converges. Let P be the set of all points in the polygon produced by the original set of points. We show that $\pi(R_{i+1}) \leq C \cdot \pi(R_i)$ for all i, where $0 < C < 1$ and $\pi(P)$ denotes Euclidean perimeter of polygon P. To show this, we need $\frac{d(p,\text{mid}(p,q))}{d(p,q)} < M < 1$ where d denotes Euclidean distance. Let $M = \max(\frac{d(p,\text{mid}(p,q))}{d(p,q)})$ for all p, q on or inside P; then in the worst-case, $\frac{d(p,\text{mid}(p,q))}{d(p,q)}$ taking the value of M for every p, q used in the rose would still cause $\pi(R)$ to scale down by a constant factor of C every iteration.

Now to see why there must exist some $M < 1$, consider a pair of points p, q in Ω, with distances from Ω to p to $\text{mid}(p,q)$ to q to Ω being a, b, c, d respectively. Then we have that

$$\frac{a}{a+b} \cdot \frac{c+d}{b+c+d} = \frac{d}{c+d} \cdot \frac{a+b}{a+b+c}.$$

Dividing all lengths be c, letting $\alpha = \frac{a}{c}, \beta = \frac{b}{c}, \delta = \frac{d}{c}$ and substituting yields

$$\begin{aligned}
\frac{\alpha}{\alpha+\beta} \cdot \frac{1+\delta}{\beta+1+\delta} &= \frac{\delta}{1+\delta} \cdot \frac{\alpha+\beta}{\alpha+\beta+1} \\
\alpha(1+\delta)^2(\alpha+\beta+1) &= \delta(\alpha+\beta)^2(\beta+1+\delta) \\
(\alpha^2+\alpha\beta+\alpha)(1+2\delta+\delta^2) &= (\delta^2+\beta\delta+\delta)(\alpha^2+2\alpha\beta+\beta^2) \\
\alpha^2+\alpha^2\delta+\alpha\beta+\alpha+2\alpha\delta &= \beta^2\delta^2+\alpha\beta\delta^2+\alpha^2\beta\delta+2\alpha\beta^2\delta+\beta^3\delta+\beta^2\delta \\
\alpha+\alpha^2+2\alpha\delta+\alpha^2\delta+\delta^2 &= \beta^3\delta+\beta^2(\delta^2+\delta+2\alpha\delta)+\beta(\alpha\delta^2+\alpha^2\delta-\alpha)
\end{aligned}$$

Solving for β, the only positive solution is

$$\beta = \sqrt{\frac{\alpha^2}{4}+\alpha+\frac{\alpha}{\delta}} - \sqrt{\frac{\alpha^2}{4}}.$$

Notice this is a monotone increasing function that converges to $1+\frac{1}{\delta}$ as $\alpha \to \infty$. $\max(\beta)$ occurs when α maximized and δ minimized; $\min(\beta)$ occurs when α minimized and δ maximized. We know there exist minimum and maximum α to δ ratios: let $j = \min(\frac{\alpha}{\delta})$ and $h = \max(\frac{\alpha}{\delta})$, and also let $l = \min(\alpha)$. Thus we find that

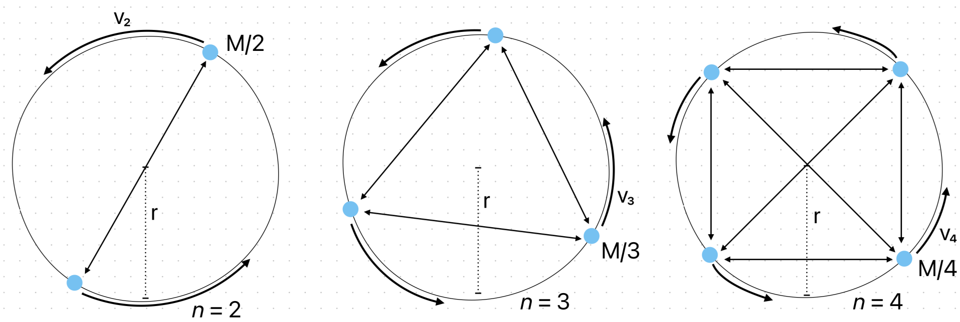
$$\max(\beta) = 1+h; \qquad \min(\beta) = \sqrt{\frac{l^2}{4}+l+j} - \sqrt{\frac{l^2}{4}}.$$

Thus $M = \frac{\max(\beta)}{\max(\beta)+1}$ exists and is less than 1.

Now for all rose iterations, we thus have $\frac{d(p,\text{mid}(p,q))}{d(p,q)} < M < 1$, and so $\pi(R_{i+1}) \leq C \cdot \pi(R_i)$ (for some $C < 1$, determined by M and P). So $\lim_{i\to\infty} \pi(R_i) = \lim_{i\to\infty} C^i \cdot \pi(R_0) = 0$, i.e. the perimeter converges to 0, making R_∞ is a single point. □

11 The n-body problem in two dimensions.

We consider a specific (two-dimensional) instance of the n-body problem, in which a total mass of M is distributed evenly across three stars, which are



all equally spaced around a ring of unchanging radius r, orbiting each other. See middle figure.

In the $n = 3$ case, the distance between two bodies in terms of r is just $d_3 = r\sqrt{3}$, with each mass being $m = M/3$. So the force of gravity between any two masses is given by

$$\frac{Gm^2}{d^2} = \frac{GM^2/9}{3r^2} = \frac{GM^2}{27r^2}.$$

Since there are three total masses, there will be two force vectors per mass. These vectors sum to a single force vector directed toward the center of the system, with magnitude equal to $\frac{\sqrt{3}GM^2}{27r^2}$.

Now since the centripetal force required to keep a mass-m object in an orbit of radius r is $\frac{mv^2}{r}$, where v is the orbital speed, we have that

$$\frac{Mv_3^2}{3r} = \frac{\sqrt{3}GM^2}{27r^2} \quad \text{or} \quad v_3 = \sqrt{\frac{\sqrt{3}GM}{9r}}.$$

Now we consider the same scenario, but for 4 identical stars orbiting each other around the same ring, again with the total mass of the system being M. Since for each star there are two stars of distance $r\sqrt{2}$ away and one of distance $2r$ away, we have that the total force of gravity towards the center for each mass is

$$\frac{\sqrt{2}GM^2}{32r^2} + \frac{GM^2}{64r^2} = \frac{GM^2}{64r^2}(2\sqrt{2}+1),$$

So

$$\frac{Mv_4^2}{4r} = \frac{GM^2}{64r^2}(1+2\sqrt{2}) \quad \text{or} \quad v_4 = \sqrt{\frac{GM}{16r}(2\sqrt{2}+1)}.$$

Now we consider the general case, i.e. for n bodies equally spaced around the ring. Labelling the chosen mass as body 0, the other $n-1$ bodies are at angles of

$$\theta_k = \frac{2\pi k}{n}, \quad k = 1, 2, \ldots n-1,$$

with the distance from body 0 to body k being

$$d_k = 2r \sin\left(\frac{\theta_k}{2}\right) = 2r \sin\left(\frac{\pi k}{n}\right).$$

So the magnitude of the gravitational force from the kth body is

$$\frac{Gm^2}{4r^2 \sin^2\left(\frac{\pi k}{n}\right)};$$

but since the radial component of this is $\frac{Gm^2 \cos\left(\frac{\pi k}{n}\right)}{4r^2 \sin^2\left(\frac{\pi k}{n}\right)}$, total radial force is

$$\frac{GM^2}{4n^2r^2} \sum_{k=1}^{n-1} \frac{\cos\left(\frac{\pi k}{n}\right)}{\sin^2\left(\frac{\pi k}{n}\right)}.$$

Centripetal force is $\frac{Mv^2}{nr}$, so we have that

$$v_n = \sqrt{\frac{GM}{4nr} \sum_{k=1}^{n-1} \frac{\cos\left(\frac{\pi k}{n}\right)}{\sin^2\left(\frac{\pi k}{n}\right)}}.$$

Now we consider what happens as n goes to infinity. Notice that for small k, we observe that $\sin(\frac{\pi k}{n}) \approx \frac{\pi k}{n}$ and $\cos(\frac{\pi k}{n}) \approx 1$: so the summand behaves like $\frac{1}{(\pi k/n)^2} = \frac{n^2}{\pi^2 k^2}$, i.e. for $S_n = \sum_{k=1}^{n-1} \frac{\cos(\pi k/n)}{\sin^2(\pi k/n)}$ we get $S_n \sim \frac{n^2}{\pi^2} \sum_{k=1}^{n-1} \frac{1}{k^2} \to \frac{n^2}{\pi^2}\frac{\pi^2}{6} = \frac{n^2}{6}$. So

$$v_n \sim \sqrt{\frac{GM}{4nr}}\sqrt{n}, \quad \text{and} \quad \lim_{n\to\infty} v_n = \infty.$$